JOURNAL OF ALGEBRA 115, 182–199 (1988)

The Maximal Subgroups of the Steinberg Triality Groups $^3D_4(q)$ and of Their Automorphism Groups

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Communicated by Walter Feit

Received September 25, 1985

1. INTRODUCTION

Throughout this paper, H_0 denotes the finite simple Steinberg triality group $^3D_4(q)$ of order $q^{12}(q^6-1)^2(q^4-q^2+1)$, where $q=p^n$ and p is prime. We define $H_1=\mathrm{Aut}(H_0)$ and we let H be any group with socle H_0. Thus

$$^3D_4(q)\cong H_0\leqslant H\leqslant H_1\cong \mathrm{Aut}(^3D_4(q)). \tag{1}$$

In Section 2 we describe elements $s_2, s_4, s_9\in H_0$ and subgroups $T_3, T_4, T_5, F\leqslant H_0$, and in Section 3 we describe elements $g_1, g_2, \phi_\alpha\in H_1\backslash H_0$ such that the following holds:

THEOREM. *Let H be as in (1) and assume that M is a maximal subgroup of H not containing H_0. Then $M_0=M\cap H_0$ is H_0-conjugate to one of the following groups:*

Group	Structure	Remarks
P_a	$[q^9]:(SL_2(q^3)\circ(\mathbf{Z}_{q-1}))\cdot d$	parabolic, $d=(2,q-1)$
P_b	$[q^{11}]:((\mathbf{Z}_{q^3-1})\circ SL_2(q))\cdot d$	parabolic, $d=(2,q-1)$
$C_{H_0}(g_1)$	$G_2(q)$	
$C_{H_0}(g_2)$	$PGL_3^\varepsilon(q)$	$2<q\equiv\varepsilon 1 \pmod 3$, $\varepsilon=\pm$
$C_{H_0}(\phi_\alpha)$	$^3D_4(q_0)$	$q=q_0^\alpha$, α prime, $\alpha\neq 3$
$N_{H_0}(F)$	$L_2(q^3)\times L_2(q)$	$p=2$, $F\cong L_2(q)$ a fundamental subgroup
$C_{H_0}(s_2)$	$(SL_2(q^3)\circ SL_2(q))\cdot 2$	p odd, involution centralizer

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0021-8693/88 $3.00

Group	Structure	Remarks
$N_{H_0}(\langle s_4\rangle)$	$((\mathbf{Z}_{q^2+q+1})\circ SL_3(q))\cdot f_+\cdot 2$	$f_+=(3, q^2+q+1)$
$N_{H_0}(\langle s_9\rangle)$	$((\mathbf{Z}_{q^2-q+1})\circ SU_3(q))\cdot f_-\cdot 2$	$f_-=(3, q^2-q+1)$
$N_{H_0}(T_3)$	$(\mathbf{Z}_{q^2+q+1})^2\cdot SL_2(3)$	
$N_{H_0}(T_4)$	$(\mathbf{Z}_{q^2-q+1})^2\cdot SL_2(3)$	
$N_{H_0}(T_5)$	$(\mathbf{Z}_{q^4-q^2+1})\cdot 4$	

Conversely, if $L\leqslant H_0$ is conjugate to one of these groups, then $N_H(L)$ is maximal in H.

Here we are using the following notational conventions. If X and Y are groups, then $X\cdot Y$ denotes an extension of X by Y and $X\circ Y$ a central product of X and Y. We write $\mathbf{Z}_m$ or simply m for the cyclic group of order m, while $[m]$ denotes an arbitrary group of that order. Also, $L_m^\varepsilon(q)$ denotes $L_m(q)$ or $U_m(q)$, according to whether ε is $+$ or $-$.

Remark. It is well known that $\mathrm{Out}(H_0)\cong\mathbf{Z}_{3n}$ (recall $n=\log_p(q)$). Thus $H\cong H_0\cdot\mathbf{Z}_m$, where $m=|H:H_0|$. In the event that $m>1$, the theorem implies that the maximal subgroups of H have shape $H_0\cdot m_0$ (m/m_0 prime), or $L\cdot m$, where L is a subgroup of H_0 occurring in the theorem. Further, any two maximal subgroups of H which are isomorphic are also conjugate in H.

Strategy of the proof. Throughout this paper, M denotes a maximal subgroup of H not containing H_0 and $M_0=M\cap H_0$. Define *The List* to be the list of groups occurring in the statement of our theorem. Thus our goals are to show that M_0 is H_0-conjugate to a group in The List and that no group in The List is contained in any other. The latter goal is rather easy, and we make some relevant remarks in Section 4. To achieve the former goal, we first observe that Thompson's theorems on groups admitting a fixed-point-free automorphism of prime order implies that $M_0\neq 1$. The case where M_0 is local is handled in Section 2 using results in [2] and [5]. The bulk of our work comes in Section 3 where we treat the case in which M_0 is non-local. For this, we regard H_0 as the centralizer in $G_0\cong P\Omega_8^+(q^3)$ of a graph-field triality automorphism of order 3. We then consider the representation of M_0 on the natural 8-dimensional projective module V for G_0 over $\mathbf{F}=GF(q^3)$. By exploiting various facts about the orthogonal geometry (V, G_0) and by appealing to several results in [1] and [8], we reduce to the case in which M_0 satisfies the conditions given on p. 469 of [1]. Namely, the socle S of M_0 is a non-abelian simple group and the representation of the preimage of S in $GL(V)$ is absolutely irreducible and is writable over no proper subfield of $\mathbf{F}$. Finally, we invoke the classification of finite simple groups to eliminate all possibilities for S.

We conclude this section with some elementary yet useful observations which appear in Section 1.3 of [8].

LEMMA 1.1. (i) *If* $N \leqslant M_0$ *and* $1 \neq N \trianglelefteq M$, *then* $M_0 = N_{H_0}(N)$.

(ii) *If* M_0 *has a non-trival normal Sylow r-subgroup for some prime r, then* M_0 *is a Sylow r-normalizer in* H_0.

(iii) *If* $M_0 \leqslant L < H_0$ *and* $O_r(M_0) \neq 1$ *for some prime r, then* $O_r(L) \leqslant O_r(M_0)$.

2. THE CASE M_0 LOCAL

The information in Tables I and II below is taken from [5].

Maximal tori. There are just seven classes of maximal tori in H_0 with representatives T_j, $0 \leqslant j \leqslant 6$. The structure of T_j and $N_{H_0}(T_j)/T_j$ is presented in Table I (see Proposition 2.3 of [5]).

Semisimple centralizers. Observe that there is an equivalence relation on the set of semisimple elements of H_0; two semisimple elements $x, y \in H_0$ are equivalent if and only if $C_{H_0}(x)$ is H_0-conjugate to $C_{H_0}(y)$. We write $[x]$ for the equivalence class containing x, and following the notation of Theorem 3.2 and Proposition 3.3 of [5], we choose representatives $s_1, ..., s_{15} \in H_0$ of the equivalence classes. In Table II we give the structure of $C_{H_0}(s_i)$ for each i.

Recall from the statement of the Theorem that $d = (2, q-1)$ and $f_\varepsilon = (3, q^2 + \varepsilon q + 1)$ for $\varepsilon = \pm$.

LEMMA 2.1. *If* $T \trianglelefteq M$ *for some maxomal torus* T *of* H_0, *then* M_0 *appears in The List.*

Proof. The group T is H_0-conjugate to T_i for some $i \in \{0, ..., 6\}$, and by Lemma 1.1(i) we have $M_0 = N_{H_0}(T)$. If $i \in \{3, 4, 5\}$, then M_0 appears on The List, as desired, so it remains to eliminate the case in which

TABLE I

T_j	Structure	$N_{H_0}(T_j)/T_j$
T_0	$\mathbf{Z}_{q^3-1} \times \mathbf{Z}_{q-1}$	D_{12}
T_1	$\mathbf{Z}_{(q^3-1)(q+1)}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$
T_2	$\mathbf{Z}_{(q^3+1)(q-1)}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$
T_3	$\mathbf{Z}_{q^2+q+1} \times \mathbf{Z}_{q^2+q+1}$	$SL_2(3)$
T_4	$\mathbf{Z}_{q^2-q+1} \times \mathbf{Z}_{q^2-q+1}$	$SL_2(3)$
T_5	$\mathbf{Z}_{q^4-q^2+1}$	$\mathbf{Z}_4$
T_6	$\mathbf{Z}_{q^3+1} \times \mathbf{Z}_{q+1}$	D_{12}

TABLE II

s_i	Structure of $C_{H_0}(s_i)$	Conditions
s_1	H_0	
s_2	$(SL_2(q^3)\circ SL_2(q))\cdot 2$	q odd
s_3	$(SL_2(q^3)\circ(\mathbf{Z}_{q-1}))\cdot d$	$q\geqslant 4$
s_4	$(\mathbf{Z}_{q^2+q+1})\circ SL_3(q))\cdot f_+$	
s_5	$((\mathbf{Z}_{q^3-1})\circ SL_2(q))\cdot d$	$q\geqslant 3$
s_6	T_0	$q\geqslant 3$
s_7	$((SL_2(q^3)\circ(\mathbf{Z}_{q+1}))\cdot d$	
s_8	T_1	
s_9	$((\mathbf{Z}_{q^2-q+1})\circ(SU_3(q))\cdot f_-$	
s_{10}	$((\mathbf{Z}_{q^3+1})\circ(SL_2(q))\cdot d$	
s_{11}	T_2	$q\geqslant 3$
s_{12}	T_3	
s_{13}	T_4	$q\geqslant 3$
s_{14}	T_5	
s_{15}	T_6	$q\geqslant 3$

$i\in\{0, 1, 2, 6\}$. Suppose first that $i\in\{0, 1\}$. Clearly there is a prime divisor s of q^3-1 which does not divide q^2-1. Thus $s\geqslant 5$ and M_0 has a normal Sylow s-subgroup. But $N_{H_0}(T)$ does not contain a Sylow s-subgroup of H_0, contrary to Lemma 1.1(ii). A similar argument shows that $i\notin\{2, 6\}$, except possibly when $i=6$ and $q=2$. In this case $M_0\cong(9\times 3).\ D_{12}$, where the D_{12} acts faithfully on the normal 3×3. Plainly M_0 normalizes a Sylow 3-subgroup of H_0, and this Sylow 3-subgroup has a characteristic subgroup C of order 3. Thus C is characteristic in M_0, and so $M_0=N_{H_0}(C)$ by Lemma 1.1(i). However, this implies that $C_{H_0}(C)$ has order $|M_0|=3^4\cdot 2^2$ or $\frac{1}{2}|M_0|=3^4\cdot 2$, yet none of the centralizers in Table II have these orders. The Lemma now follows. ∎

LEMMA 2.2. *Suppose that $\langle x\rangle\trianglelefteq M$ for some $x\in H_0$ of prime order r, with $r\neq p$. Then M_0 appears in The List.*

Proof. By Lemma 1.1(i), $M_0=N_{H_0}(\langle x\rangle)$ and we also observe that $C=C_{H_0}(x)\trianglelefteq M$. Now $x\in[s_i]$ for some $i\geqslant 2$, and if $i\in\{6, 8, 11, 12, 13, 14, 15\}$ then we may appeal to Lemma 2.1. Also if $i\in\{2, 4, 9\}$ then M_0 does indeed appear on The List. So it remains to eliminate the case $i\in\{3, 5, 7, 10\}$. Suppose first that $i\in\{5, 10\}$, so that $C\cong((\mathbf{Z}_{q^3+\varepsilon 1})\circ SL_2(q))\cdot d$, where $\varepsilon=\pm$. If $(q, \varepsilon)=(2, +)$, then $M_0\cong(9\times L_2(2))\cdot 2$, which has a normal Sylow 3-subgroup of order 27, against Lemma 1.1(ii). So it may be assumed that $(q, \varepsilon)\neq(2, +)$, and thus there is an element y in the $\mathbf{Z}_{q^3+\varepsilon 1}$ of prime order s, where s does not divide $|SL_2(q)|$. Thus C has a normal cyclic Sylow s-subgroup. Now $C=C_{H_0}(y)$ and $M_0\leqslant N_{H_0}(\langle y\rangle)$, and thus $|M_0 : C| \leqslant |\mathrm{Aut}(\langle y \rangle)| < s$, which means M_0 also has a normal Sylow s-subgroup. However, because $(q^3 + \varepsilon 1)^2 \mid |H_0|$, it follows that M_0 does not contain a Sylow s-subgroup of H_0, contradicting Lemma 1.1(ii) again.

Now assume that $i \in \{3, 7\}$, so that $C \cong (SL_2(q^3) \circ (\mathbf{Z}_{q+\varepsilon 1})) \cdot d$. It is well known that H_0 has a subgroup $K = (F \circ B) \cdot d$, where $F \cong SL_2(q)$ is a fundamental subgroup and $B \cong SL_2(q^3)$. Clearly K contains a copy of C, and since C embeds in none of the centralizers $C_{H_0}(s_j)$ for $j \notin \{1, i\}$ (see Table II), we deduce that K contains an H_0-conjugate of C. Hence without loss we can assume that $C < K$. Now K contains a unique subgroup $SL_2(q^3)$, and so $B \leqslant C$. Therefore B is the last term in the derived series of M_0, which means $M_0 = N_{H_0}(B)$ by Lemma 1.1(i). Thus because $B \trianglelefteq K$, we have $K \leqslant M_0 = N_{H_0}(\langle x \rangle)$, and so $\langle x \rangle \trianglelefteq K$. Now $r \geqslant 3$ (as $x \notin [s_2]$), which means K has a normal cyclic subgroup of odd prime order. Clearly this occurs only when $q = 2$ and $r = 3$. In this case, $C \cong L_2(8) \times 3$ and $K = M_0 \cong L_2(8) \times L_2(2)$, and so M_0 appears on The List, as desired. ∎

We record some further information about the local structure of H_0. Recall that for any prime r and for any group X, $m_r(X)$ is the *r-rank of X*, which is the maximal rank of an abelian r-subgroup of H_0.

LEMMA 2.3. (i) *When q is odd, there is a unique class of involutions in H_0.*

(ii) *When q is odd, $m_2(H_0) = 3$.*

(iii) *When $(3, q) = 1$, $m_3(H_0) = 2$.*

(iv) *When $(r, q) = 1$ for some prime $r \geqslant 5$, the Sylow r-subgroups of H_0 are abelian and $m_r(H_0) \leqslant 2$.*

Proof. Assertions (i) and (ii) are well known (see [6], for example), while (iii) and (iv) follow directly from 10.1 of [7]. ∎

We now prove the main result of this section.

PROPOSITION 2.4. *If M_0 is local, then M_0 is conjugate to some group in The List.*

Proof. Let E be a minimal normal, elementary abelian subgroup of M which is contained in M_0. Assume that E has order r^a, where r is prime. Thus $M_0 = N_{H_0}(E)$ by Lemma 1.1(i), and we consider the various possibilities for r and a.

Case $p = r$. Lemma 1.1(i) ensures that $M_0 = N_{H_0}(O_p(M_0))$, whence M_0 is a parabolic subgroup of H_0 by the Corollary in [2]. The proof of 1.6.1 of [8] may be applied here to show that M_0 is not a Borel subgroup of H_0, and so M_0 lies strictly between a Borel subgroup and H_0. Thus M_0 is conjugate to a group on The List.

Case $p \neq r \geqslant 3$ and $a \geqslant 2$. By Lemma 2.3, we have $a = 2$, and it follows from II.5.1 of [11] that $E \leqslant T$ for some maximal torus T of H_0. Thus T is conjugate to T_i for some $i \in \{0, 3, 4, 6\}$. Using the notation in Sect. 2 in [5], we may write elements of T as 4-tuples (z_1, z_2, z_3, z_4) with $z_i \in \bar{\mathbf{F}}^*$, where $\bar{\mathbf{F}}$ is an algebraic closure of $\mathbf{F} = GF(q^3)$. Suppose for the moment that $i = 0$. By Proposition 2.3 of [5] we see that

$$E = \{(z_1, z_2, z_1^q, z_1^{q^2}) : z_1^r = z_2^r = 1\}.$$

Let $\omega \in \bar{\mathbf{F}}^*$ be a primitive rth root of unity, and consider $e = (\omega, \omega, \omega^q, \omega^{q^2}) \in E$. Theorem 3.2 of [5] ensures that $e \in [s_6]$, and hence $C_{H_0}(e) = T$. Therefore $C_{H_0}(E) = T$. Similarly, if $i \in \{3, 4, 6\}$, then $C_{H_0}(E) = T$. Thus in all cases, $T = C_{H_0}(E)$, and hence $T \trianglelefteq M$. The result now follows from Lemma 2.1, as desired.

Case $p \neq r$ and $a = 1$. This is treated in Lemma 2.2.

Case $p \neq r = a = 2$. An inspection of the structure of the involution centralizer (see 14.1.3α, γ of [7] and Proposition 3.3 of [5]) shows that H_0 has just two classes of four groups, with centralizers $T_0 \cdot 2$ and $T_6 \cdot 2$. Since M_0 normalizes $C_{H_0}(E)$, it follows that M_0 is odd-local so we appeal to a previous case.

Case $p \neq r = 2$ and $a \geqslant 3$. Since the centralizer of a four-group in H_0 is a p'-group, $C_{H_0}(E)$ is also a p'-group. Now for $i \geqslant 3$, $m_2(C_{H_0}(s_i)) \leqslant 2$, and thus all elements in $C_{H_0}(E) \backslash 1$ have order 2. Hence $C_{H_0}(E) = E \cong 2^3$, by Lemma 2.3(ii). Let $g_1 \in H_1$ be as in (3) below, so that $C_{H_0}(g_1) \cong G_2(q)$. Since $C_{H_0}(g_1)$ contains a Sylow 2-subgroup of H_0, we may assume that $E \leqslant C_{H_0}(g_1)$. Now take $x \in N_H(E)$, and notice that since $\mathrm{Out}(H_0)$ is abelian, $g_1^x = hg_1$ for some $h \in H_0$. Clearly g_1 and g_1^x centralize E, and so $h \in C_{H_0}(E) = E \leqslant C_{H_0}(g_1)$. Therefore $[h, g_1] = 1$, and since $|hg_1| = |g_1| = 3$, it follows that $|h| \neq 2$. Therefore $h = 1$, and so $x \in C_H(g_1)$. Thus $M = N_H(E) < C_H(g_1)$, contrary to the maximality of M. The proof of the Proposition is now complete. ∎

3. The Case M_0 Non-Local

Assume M_0 is non-local, so that the socle $S = \mathrm{soc}(M_0)$ is a direct product of non-abelian simple groups. Evidently $C_{M_0}(S) = 1$, and as $M_0 = N_{H_0}(S)$ by Lemma 1.1(i), we have

$$C_{H_0}(S) = 1. \tag{2}$$

We now prove a useful result. Recall $H_1 = \mathrm{Aut}(H_0) \cong H_0 \cdot \mathbf{Z}_{3n}$, where $n = \log_p(q)$.

LEMMA 3.1. *If $C_{H_1}(S) \neq 1$, then M_0 appears on The List.*

Proof. Take $x \in C_{H_1}(S)$ of prime order. Clearly $[M, x] \leqslant C_{H_0}(S) = 1$, and hence $M = C_H(x)$. Thus $M_0 = C_{H_0}(x)$.

Suppose in this paragraph that $|x| = 3$. In the language of 9.1 of [7], x is either a field or a graph automorphism of H_0. However, x is not a field automorphism as $|{}^3D_4(q^{1/3})|$ does not divide $|H_0|$. Hence x is a graph automorphism, and according to 9.1.3 of [7], H_1 has just two conjugacy classes of subgroups of order 3 not contained in H_0, with representatives $\langle g_1 \rangle$ and $\langle g_2 \rangle$, say. We may write

$$C_{H_0}(g_1) \cong G_2(q)$$

and

$$C_{H_0}(g_2) \cong \begin{cases} PGL_3^{\varepsilon}(q) & \text{if } q \equiv \varepsilon 1 \pmod 3 \\ [q^5] \cdot SL_2(q) & \text{if } 3 \mid q. \end{cases} \tag{3}$$

Now M_0 is not $[q^5] \cdot SL_2(q)$ as M_0 is non-local. Therefore $M_0 \cong G_2(q)$ or $PGL_3^{\varepsilon}(q)$. It follows from the proof of 9.1.3 of [7] that both $G_2(q)$ and $PGL_3^{\varepsilon}(q)$ extend to groups $G_2(q) \cdot \mathbf{Z}_{3n}$ and $PGL_3^{\varepsilon}(q) \cdot \mathbf{Z}_{3n}$ in H_1. Consequently H_0 is transitive on each of these two H_1-classes of groups of order 3, which means $\langle x \rangle$ is H_0-conjugate to $\langle g_1 \rangle$ or $\langle g_2 \rangle$. Therefore M_0 is H_0-conjugate to either $C_{H_0}(g_1)$ or $C_{H_0}(g_2)$, and so appears on The List. (Note that when $M_0 \cong PGL_3^{\varepsilon}(q)$, we stipulate $2 < q$ in our Theorem, for $PGU_3(2)$ is solvable.)

Now suppose that $|x| \neq 3$. It follows from 7.2 of [7] that for each prime divisor α of n, with $\alpha \neq 3$, H_0 is transitive on groups of order α not contained in H_0. Letting $\langle \phi_\alpha \rangle$ be a representative of this H_0-orbit, M_0 is H_0-conjugate to $C_{H_0}(\phi_\alpha)$, and by 9.1.1 of [7], $C_{H_0}(\phi_\alpha) \cong {}^3D_4(q_0)$, where $q = q_0^{\alpha}$. The proof is complete. ∎

For the remainder of this section, we regard H_0 as the centralizer in $P\Omega_8^+(q^3)$ of a graph-field triality automorphism of order 3. We set the following scene.

Let V be an 8-dimensional vector space over $\mathbf{F} = GF(q^3) = GF(p^{3n})$, and let $Q\colon V \to \mathbf{F}$ be a non-degenerate quadratic form of (Witt) defect 0. We define G_0, Ω, O, and Γ as in Sect. 2 of [8]. Thus O is the orthogonal group

$$O = O(V, Q, \mathbf{F}) = \{ g \in GL(V) \colon Q(v^g) = Q(v) \text{ for all } v \in V \},$$

and $\Omega = O'$, a perfect group. Further Γ is the set of all $g \in \Gamma(V)$ such that $Q(v^g) = \lambda Q(v)^\sigma$, where $\lambda \in \mathbf{F}^*$ and $\sigma \in \mathrm{Aut}(\mathbf{F})$ are independent of v. Here $\Gamma(V)$ is the set of all non-singular $\mathbf{F}$-semilinear transformations of V. For $X \in \{\Omega, O, \Gamma\}$, we write PX for the corresponding projective group, and we let G_0 be the simple group $P\Omega \cong P\Omega_8^+(q^3)$.

We now describe the structure of the $A = \mathrm{Aut}(G_0)$ (more details can be found in Sect. 1.4 of [8]). First define D as the group of inner and diagonal automorphisms of G_0, so that

$$D \cong \begin{cases} G_0 & \text{if } q \text{ is even} \\ G_0 \cdot 2^2 & \text{if } q \text{ is odd.} \end{cases}$$

Second, by Sect. 12.2 of [3] we can choose graph automorphisms of G_0 which extend D to a group $\Theta = D \cdot S_3$ satisfying

$$\Theta \cong \begin{cases} G_0 \cdot S_3 & \text{if } q \text{ is even} \\ G_0 \cdot S_4 & \text{if } q \text{ is odd.} \end{cases}$$

Third, there exists a group Φ of field automorphisms of G_0 with $\Phi \cong \mathrm{Aut}(F) \cong Z_{3n}$. We have $A = \Theta\Phi$ and it is not difficult to show (see Sect. 1.4 of [8], for example), that Φ is central modulo G_0. Thus defining $G_1 = G_0\Phi$, we have

$$A/G_0 = \Theta/G_0 \times G_1/G_0 \cong \begin{cases} S_3 \times \mathbf{Z}_{3n} & \text{if } q \text{ is even} \\ S_4 \times \mathbf{Z}_{3n} & \text{if } q \text{ is odd.} \end{cases} \tag{4}$$

Moreover, we can choose a graph automorphism $\theta \in \Theta$ of order 3 such that $[\theta, \Phi] = 1$. And by Sect. 13.4 of [3] there is a field automorphism $\phi \in \Phi$ of order 3 such that $\tau = \theta\phi$ is a graph-field triality automorphism with $C_{G_0}(\tau) \cong {}^3D_4(q)$. Hence we may write

$$H_0 = C_{G_0}(\tau).$$

Since $[\Phi, \tau] = 1$, it follows that Φ normalizes H_0, and we claim that

$$C_{H_0\Phi}(H_0) = 1. \tag{5}$$

For suppose that $x \in C_{H_0\Phi}(H_0)$ has prime order r. Then by 7.2 and 9.1.1.d of [7], $H_0 \leqslant O^{p'}(C_{G_0}(x)) \cong P\Omega_8^+(q^{3/r})$, which is impossible by Lagrange's Theorem. Therefore (5) holds, and so $H_0\Phi$ embeds in $H_1 = \mathrm{Aut}(H_0)$. Since $|H_0\Phi| = |H_1|$, it may be assumed that

$$H_1 = H_0\Phi.$$

Thus

$$C_{G_1}(\tau) = C_{G_0}(\tau)\Phi = H_1. \tag{6}$$

In view of (4) there is a homomorphism π from A to Σ with kernel G_1, where Σ is S_3 or S_4 according to whether q is even or odd. We take π to be the homomorphism described explicitly in Sect. 1.4 of [8]. Obviously $\pi(C_A(S))$ contains the 3-cycle $\pi(\tau)=\pi(\theta)$, and this next result treats a case in which $\pi(C_A(S))$ contains more than just this 3-cycle.

LEMMA 3.2. *If $\pi(C_A(S))$ contains a 2-cycle, then M_0 appears on The List.*

Proof. Take $y \in C_A(S)$ with $\pi(y)$ a 2-cycle. If $\pi(y)$ does not normalize $\langle \pi(\tau) \rangle$, then $\langle \pi(y), \pi(\tau) \rangle = \Sigma \cong S_4$. Thus we may replace y by another suitable element of $C_A(S)$ in order to assume that $\pi(y)$ does indeed normalize $\langle \pi(\tau) \rangle$. Therefore y inverts τ modulo G_1. However, $\tau = \theta\phi$ with $\theta \in \Theta$ and $\phi \in \Phi \leqslant G_1$, and as G_1 is central modulo G_0, it follows from (4) that y inverts θ modulo G_0. Hence

$$[y, \tau]\tau \equiv [y, \theta]\,\theta\phi \equiv \phi \pmod{G_0}.$$

So defining $K = G_0\langle \phi \rangle \cong G_0 \cdot 3$, we see that $[y, \tau]\tau \in C_K(S) \backslash G_0$, and so $3 \mid |C_K(S)|$. Observe that τ normalizes K (since $[\tau, \phi]=1$). Thus τ normalizes $C_K(S)$, and so τ normalizes a Sylow 3-subgroup of $C_K(S)$. Hence τ centralizes an element x of order 3 in $C_K(S)$. Therefore $x \in C_K(\tau) \leqslant C_{G_1}(\tau) = H_1$ (see (6)), and so $x \in C_{H_1}(S)$. The result now follows from Lemma 3.1. ∎

Since the multiplier of H_0 is trivial, the preimage of H_0 in Ω (with q odd) is a group $\langle -1 \rangle \times \tilde{H}_0$, where $\tilde{H}_0 \cong H_0$. Therefore the preimage of S in Ω is a group $\langle -1 \rangle \times \tilde{S}$ with $\tilde{S} \cong S$. As a convenience we put $\tilde{S} = S$ when q is even, so that we may talk about $\tilde{S}$ for all q. In the rest of this section, we study the action of $\tilde{S}$ on V in detail.

Recall the definition of $\Gamma(V)$, Γ and $P\Gamma$ given after Lemma 3.1.

LEMMA 3.3. *If $\tilde{S}$ is absolutely irreducible on V and the representation of $\tilde{S}$ is writable over a proper subfield of* $\mathbf{F}$, *then M_0 appears on The List.*

Proof. There exists an element $\psi \in \Gamma(V) \backslash GL(V)$ such that $[\psi, \tilde{S}] = 1$, and we may choose ψ to have prime order modulo $GL(V)$. Since $\tilde{S}$ is absolutely irreducible, $\langle \psi \rangle \cap GL(V) \leqslant C_{GL(V)}(\tilde{S}) = Z(GL(V))$, and so ψ has prime order modulo $Z(GL(V))$. Further 1.7.1.i of [8] ensures that $\psi \in \Gamma$. Hence if $\bar{\psi}$ is the image of ψ in $P\Gamma$, then $\bar{\psi} \in C_{P\Gamma}(S)$ and $|\bar{\psi}|$ is prime. Since $|P\Gamma : G_1|$ divides 8, $\pi(\bar{\psi})$ has order 1 or 2. If $\pi(\bar{\psi})$ is a 2-cycle then we may appeal to Lemma 3.2. So we are left with the cases in which either $\pi(\bar{\psi}) = 1$ or $\pi(\bar{\psi})$ lies in the normal four-group of Σ. In either case, $\bar{\psi} \in D\Phi$

(see (1h) in [8]) and as Φ is central modulo G_0, we have $[A, D\Phi]=D$. Consequently

$$[\tau, \bar{\psi}] \in [C_A(S), C_{D\Phi}(S)] \leqslant C_D(S). \tag{7}$$

However, $C_D(S)=1$ by 1.2.3 of [8], and so (7) implies that $[\bar{\psi}, \tau]=1$. In particular, $[\pi(\bar{\psi}), \pi(\tau)]=1$, and hence $\pi(\bar{\psi})$ cannot be a non-trivial element in the normal four-group in Σ when q is odd. Therefore $\pi(\bar{\psi})=1$, which is to say $\bar{\psi} \in G_1$. Hence $1 \neq \bar{\psi} \in C_{G_1}(\tau)=H_1$, and the result holds in view of Lemma 3.1. ∎

Recall that a subspace W is *totally singular* if $Q(w)=0$ for all $w \in W$. When the restriction of Q to W is a non-degenerate quadratic form on W, then W is *non-degenerate*. If W is non-degenerate and $\dim(W)=m$ is even, then Q induces an O_m^{ε}-geometry on W, where $\varepsilon = \pm$. In this case we call W an *εm-space*. A vector $v \in V$ is *non-singular* if $Q(v) \neq 0$. Also if $x \in O$, then we write $\bar{x}$ for the image of x in $PO=O/Z(O)$.

LEMMA 3.4. *If S fixes a non-singular 1-space or a non-degenerate 2-space or a nondegenerate 3-space in V, then M_0 appears on The List.*

Proof. If S fixes the non-singular 1-space $\langle v \rangle$ ($v \in V$), then S centralizes the reflection $\bar{r}_v \in PO$. (Here r_v is the reflection in v.) But $\pi(\bar{r}_v)$ is a 2-cycle (see assertion (1i) in [8]), so the result follows from Lemma 3.2. Now suppose that S fixes the non-degenerate 2-space W. Since $O_2^{\varepsilon}(q^3)$ is solvable while $\tilde{S}$ is perfect, $\tilde{S} \leqslant C_{\Omega}(W)$. But W contains a non-singular vector, and we reduce to the previous case. Finally, if W is an S-invariant non-degenerate 3-space, then $\tilde{S}$ centralizes the involution z which acts as -1 on W and $+1$ on $W^{\perp}$. However, $\bar{z} \in PO \backslash PSO$, which means $\pi(z)$ is a 2-cycle, and we appeal to Lemma 3.2 again. ∎

Hereafter we can assume that

$$\begin{array}{l} S \text{ does not fix a non-singular 1-space,} \\ \text{a non-degenerate 2-space, or a non-degenerate 3-space.} \end{array} \tag{8}$$

In the next three lemmas, we consider the case where S fixes a totally singular subspace of V. Our arguments depend on some facts relating the totally singular subspaces of V to the parabolic subgroups of G_0. We digress briefly to discuss these facts. For a more detailed discussion see Sect. 1.6 in [8].

Label the four nodes of the Dynkin diagram of G_0 as 1, 2, 3, 4 with 2 the central node, and let $P_{i,j,\ldots}$ be the parabolic subgroup of G_0 corresponding to the set of nodes $\{i, j, \ldots\}$. Any triality automorphism cyclically permutes the nodes 1, 3, 4, and so the only maximal parabolics which are normalized by a triality automorphism are G_0-conjugate to $P_{1,3,4}$. These are the stabilizers in G_0 of totally singular 2-spaces (or totally singular *lines*). Relabelling the nodes if necessary, we can assume that the maximal parabolic $P_{1,2,4}$ is the stabilizer of a totally singular 1-space (totally singular *point*), while $P_{1,3,4}$ and $P_{2,3,4}$ stabilize totally singular 4-spaces (totally singular *solids*). Thus G_0 has just two orbits $\mathscr{S}_1$, $\mathscr{S}_2$ on totally singular solids, and two such solids lie in the same orbit if and only if their intersection has even dimension. Therefore each totally singular 3-space (totally singular *plane*) lies in exactly two totally singular solids (one from each $\mathscr{S}_i$), and so the stabilizer in G_0 of a totally singular plane also stabilizes the two totally singular solids which contain it. The action of A on the maximal parabolic subgroups of G_0 induces an action on the set of totally singular points, lines and solids. Namely, if U is a totally singular point, line or solid, and $x \in A$, then U^x is defined by $N_{G_0}(U^x) = (N_{G_0}(U))^x$. Thus if U is a totally singular line, then so is U^x. This action preserves the following incidence relation: two totally singular subspaces are *incident* if and only if one contains the other or if they are a pair of totally singular solids which intersect in a plane. Thus, for example, if U is a totally singular point, then U^τ and U^{τ^2} are totally solids in $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, and $\dim(U^\tau \cap U^{\tau^2})$ is 3 or 1 according to whether U is or is not contained in U^τ.

LEMMA 3.5. *The group $M \times \langle \tau \rangle$ does not normalize a proper parabolic subgroup of G_0.*

Proof. Otherwise $M \times \langle \tau \rangle$ normalizes a parabolic P which is either a Borel subgroup or else is G_0-conjugate to P_2 or $P_{1,3,4}$. The maximality of M yields $M = N_H(P)$, giving $M_0 = N_{H_0}(P) = P \cap H_0 = C_P(\tau)$. So clearly P is not one of the Borel subgroups, for these are solvable while S is not. Hence if R is the solvable radical of P (that is, the largest normal solvable subgroup of P), then

$$|R| = \begin{cases} \dfrac{1}{d^2} q^{33}(q^3-1)^3 & \text{if} \quad P \cong P_2 \\[2ex] \dfrac{1}{d^2} q^{27}(q^3-1) & \text{if} \quad P \cong P_{1,3,4}. \end{cases}$$

(The structure of P_2 and $P_{1,3,4}$ is given in [8].) In particular, $|R| \equiv 0$ or 2 (mod 3), which means τ cannot act fixed-point-freely on R. Thus $1 \neq C_R(\tau) \trianglelefteq M_0$, contrary to the fact that M_0 is non-local. ∎

LEMMA 3.6. *The group S does not fix a totally singular point, plane, or solid.*

Proof. Since the stabilizer of a totally singular plane is contained in the stabilizer of a totally singular solid, and since τ permutes the totally singular solids and the totally singular points as described above, it suffices to prove that S does not fix a totally singular point. Thus we suppose for a contradiction that $S \leqslant N_{G_0}(\langle v \rangle)$ for some totally singular vector $v \neq 0$. We argue that

$$M \text{ fixes a totally singular point.} \tag{9}$$

Clearly S fixes $\langle v^m \rangle$ for all $m \in M$, and if $(v, v^m) \neq 0$, then S fixes the non-degenerate 2-space $\langle v, v^m \rangle$, contrary to (8). Consequently $W = \langle v \rangle^M$ is totally singular. Also $\tilde{S}$ acts as a group of diagonal matrices on W, and so as $\tilde{S}$ is perfect, we have

$$\tilde{S} \leqslant C_\Omega(W). \tag{10}$$

Since the representation of $\tilde{S}$ on W is dual to its representation on $V/W^\perp$, we also have

$$\tilde{S} \leqslant C_\Omega(V/W^\perp). \tag{11}$$

Clearly (10) and (11) imply that $W \neq W^\perp$, which means $\dim(W) < 4$. If $\dim(W) = 3$, then $W^\perp/W$ inherits on O_2^+-geometry from Q, and thus $\tilde{S}$ embeds in the dihedral group $O_2^+(q^3)$, a contradiction. This leaves the case in which $\dim(W) \leqslant 2$, and if $\dim(W) = 1$ then obviously (9) holds. We may assume therefore that $\dim(W) = 2$. Thus W is not τ-invariant by Lemma 3.5, and so $U = W^\tau \neq W$. If either $W \cap U$ or $W \cap U^\perp$ has dimension 1 then (9) holds, and if $W \leqslant U^\perp$, then M fixes the totally singular solid $W \oplus U$. But then $(W \oplus U)^\tau$ or $(W \oplus U)^{\tau^2}$ is an M-invariant totally singular point and again (9) holds. So we are left with the case $W \cap U = W \cap U^\perp = 0$, and this means that $W \oplus U$ is a $+4$-space. But then by (10) and (11) we have $\tilde{S} \leqslant C_\Omega(W \oplus U)$, against (8). Thus we have established (9).

There is no harm in redefining $\langle v \rangle$ as the point provided by (9). Clearly $M \times \langle \tau \rangle$ normalizes $K = N_{G_0}(\langle v \rangle, V_1, V_2)$, where $V_i = \langle v \rangle^{\tau^i} \in \mathscr{S}_i$ for $i = 1, 2$. Also observe that $S \leqslant K$. Suppose for the moment that $v \in V_1$. Then as τ preserves incidence, $\dim(V_1 \cap V_2) = 3$ and $v \in V_2$. Hence by Sect. 1.6 of [8], K is a parabolic of G_0 (corresponding to the central node of the Dynkin diagram), and we have contradicted Lemma 3.5. This leaves the case $v \notin V_1$. But then the proof of 4.1.4 (Step 1) of [8] shows that K stabilizes a flag, forcing S to be contained in a Borel subgroup. However, this is impossible, as the Borel subgroups are solvable. This final contradiction finishes the proof. ∎

LEMMA 3.7. *If $\tilde{S}$ fixes a unique pair of orthogonal $+4$-spaces then M_0 occurs in The List.*

Proof. Write $S \leqslant L = N_{G_0}\{W, W^\perp\}$, where W and $W^\perp$ are $+4$-spaces. If $x \in M \times \langle \tau \rangle$, then obviously $S \leqslant L^x$. However, by 15.1.7 of [1], L^x is the normalizer in G_0 of another pair of orthogonal $+4$-spaces, so our assumption about uniqueness ensures $L^x = L$. Thus $M \times \langle \tau \rangle \leqslant N_A(L)$, and as in the proof of Lemma 3.5, the maximality of M yields $M_0 = C_L(\tau)$. If q is odd, then $M \times \langle \tau \rangle$ centralizes the involution $z \in Z(L)$, which means $z \in Z(M_0)$, against (2). Therefore q is even and $L = N \cdot 2^2$, where N is a direct product of four components $L_2(q^3)$, permuted regularly by L/N. Suppose for the moment that τ normalizes each component. Then τ centralizes an element of order 3 in each component, which means $m_3(C_N(\tau)) \geqslant 4$, against Lemma 2.3(iii). Therefore τ cyclically permutes three of these components, and so M_0 contains a subgroup $L_2(q^3)$. If τ induces an inner automorphism on the fourth component, then M_0 contains $\mathbf{Z}_{q^3 \pm 1} \times L_2(q^3)$, contrary to Table II. Therefore τ induces an outer automorphism on the fourth component. Hence by 9.1.1 of [7], the centralizer of τ in this fourth component is a group $L_2(q)$, and so $L_2(q^3) \times L_2(q) \cong C_N(\tau) \trianglelefteq M_0$. Since $L\langle \tau \rangle / N \cong A_4$, it follows that $C_L(\tau) = C_N(\tau)$. Thus to complete the proof it suffices to show that H_0 contains a unique class of groups $L_2(q) \times L_2(q^3)$. Suppose therefore that $J, K \leqslant H_0$ and that $J \cong K \cong L_2(q^3) \times L_2(q)$. By Table II, H_0 contains a unique class of groups $L_2(q^3) \times \mathbf{Z}_{q+1}$, so we can assume that $L_2(q^3) \times \mathbf{Z}_{q+1} \leqslant J \cap K$. Hence J and K contain the *same* group $L_2(q^3)$. Now take a subgroup $\mathbf{Z}_{q^3-1}$ in this $L_2(q^3)$, and note that this $\mathbf{Z}_{q^3-1}$ is centralized by an $L_2(q)$ in each of J and K. Clearly elements in the $\mathbf{Z}_{q^3-1}$ lie in $[s_5]$, and hence they are centralized by a unique $L_2(q)$. Therefore J and K contain the same $\mathbf{Z}_{q^3-1} \times L_2(q)$, and it follows that $J = K$, as desired. ∎

LEMMA 3.8. *If S fixes a totally singular line, then M_0 appears in The List.*

Proof. Suppose that S fixes the totally singular line W. By Lemma 3.5, W is not $M \times \langle \tau \rangle$-invariant, and so S fixes a totally singular line $U = W^x \neq W$ for some $x \in M \times \langle \tau \rangle$. Lemma 3.6 ensures that $W \cap U = W \cap U^\perp = 0$, and so $W \oplus U$ is a $+4$-space. Hence S acts on the $+4$-space $Y = (W \oplus U)^\perp$, and by (8) and Lemma 3.6, S does not fix a 1-space or a non-degenerate 2-space in Y. Also, if S fixes a totally singular line Y_1 in Y, then S fixes the solid $W \oplus Y_1$, contrary to Lemma 3.6. Therefore S acts irreducibly on Y, and so Y is the *unique* 4-space on which S acts irreducibly. Thus S fixes a unique pair of orthogonal $+4$-spaces, and we can appeal to Lemma 3.7. ∎

Hereafter we can assume that

$$S \text{ does not fix a totally singular line.} \tag{12}$$

As a convenience in the proof of this next result, we write

$$\tilde{S}(U) = \tilde{S}/C_{\tilde{S}}(U),$$

where U is any S-invariant subspace of V, so that $\tilde{S}(U)$ embeds in $GL(U)$.

LEMMA 3.9. *If S fixes a non-degenerate 4-space, then M_0 appears in The List.*

Proof. Write $S \leqslant L = N_{G_0}\{U, W\}$, where U and W form a pair of orthogonal ε4-spaces ($\varepsilon = \pm$). Assertions (8), (12), and Lemma 3.6 ensure that $\tilde{S}$ is irreducible on U and W. We claim that

$$\tilde{S} \text{ is absolutely irreducible on } U \text{ and } W. \tag{13}$$

If $\tilde{S}$ fails to be absolutely irreducible on U, then by 7.6 of [1], $\tilde{S}(U)$ embeds in a group $O_2^{\varepsilon}(q^6)$ or $GU_2(q^3)$. In the first case $\tilde{S}(U)$ is solvable, which means $\tilde{S}(U) = 1$, a contradiction. So assume that $\tilde{S}(U)$ embeds in $GU_2(q^3)$. Since $\tilde{S}(U) \neq 1$, and because no non-abelian simple group has a faithful representation of degree 2 in odd characteristic, we conclude that q is even. Furthermore, $\tilde{S}(U)$ actually embeds in $U_2(q^3) \cong L_2(q^3)$, and the only non-solvable subgroups of $L_2(q^3)$ are groups $L_2(q_0)$, where $GF(q_0)$ is a subfield of $\mathbf{F}$. It follows that $\mathbf{F}$ is a splitting field for $\tilde{S}(U)$, and this contradicts the fact that $\tilde{S}(U)$ is irreducible yet not absolutely irreducible. Thus $\tilde{S}$ is absolutely irreducible on U and similarly on W, proving (13).

We now argue that

$$U \not\cong W \text{ as } \tilde{S}\text{-modules.} \tag{14}$$

Assume for a contradiction that $U \cong W$ as $\tilde{S}$-modules. Thus there exists bases $(u_1, ..., u_4)$ and $(w_1, ..., w_4)$ of U and W, respectively, such that elements in $\tilde{S}$ have the form $\left(\begin{smallmatrix} g & 0 \\ 0 & g \end{smallmatrix}\right)$ $(g \in GL_4(q^3))$ with respect to the basis $(u_1, ..., u_4, w_1, ..., w_4)$ of V. As in the proof of 4.1.7 (Step 5) in [8], we write $h \otimes g$ for the matrix

$$\begin{pmatrix} xg & yg \\ zg & wg \end{pmatrix} \in GL(V),$$

where $g \in GL_4(q^3)$ and $h = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in GL_2(q^3)$. With this convention, $C_{GL(V)}(\tilde{S}) = GL_2(q^3) \otimes 1$. Moreover, by (13) and the proof of 1.7.1.i of [8], there exists $\lambda \in \mathbf{F}^*$ such that

$$Q(w_i) = \lambda Q(u_i) \text{ for all } i. \tag{15}$$

If $-\lambda^{-1} = \mu^2$ for some $\mu \in \mathbf{F}$, then S fixes the totally singular solid $\langle u_i + \mu w_i : i \leqslant 4 \rangle$, against Lemma 3.6. Therefore $-\lambda$ is a non-square, and so q is odd. Let B be the matrix of the bilinear form $(\, , \,)$ on U with respect to $(u_1, ..., u_4)$. By (15), the matrix of $(\, , \,)$ on V with respect to $(u_1, ..., u_4, w_1, ..., w_4)$ is $b \otimes B$, where $b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right) \in GL_2(q^3)$. Thus $C_O(\tilde{S}) = \{h \otimes 1 \in GL_2(q^3) \otimes 1 : h^t bh = b\}$ (here t denotes transpose). Since $-\lambda$ is a non-square, 1.2.1 of [8] implies that b is the matrix of a non-degenerate symmetric bilinear form giving rise to an O_2^--geometry on a 2-space, and hence $C_O(\tilde{S}) \cong O_2^-(q^3) \cong D_{2(q^3+1)}$. Thus Ω has a subgroup $K = \Omega_2^-(q^3) \otimes \tilde{S}$, and since $\Omega_2^-(q^3) \cong \mathbf{Z}_{q^3+1}$ is irreducible in $GL_2(q^3)$, K is irreducible on V. However, $\Omega_2^-(q^3) = Z(K)$, hence 7.6 and 7.7 of [1] imply that K is contained in an irreducible but not absolutely irreducible copy of $GU_4(q)$ in O. Thus by 15.1.5 of [1], S is contained in the stabilizer of -2-space, contradicting (8). Thus we have established (14).

It now follows that U and W are the *only* 4-spaces fixed by S. Suppose for the moment that U and W are -4-spaces. Then by 15.1.8 of [1], $\mathrm{Hom}_{\tilde{S}}(V)$ contains a quadratic field extension $\mathbf{E}$ of $\mathbf{F}$ and $\mathbf{Z}_{q^6-1} \cong \mathbf{E}^* \leqslant C_{GL(V)}(\tilde{S})$. On the other hand, (13) and (14) imply that $C_{GL(V)}(\tilde{S})$ fixes U and W and induces scalars on these spaces; thus $C_{GL(V)}(\tilde{S}) \cong \mathbf{Z}_{q^3-1} \times \mathbf{Z}_{q^3-1}$. This contradiction forces U and W to be $+4$-spaces, and the result now follows from Lemma 3.7. ∎

Assertions (8) and (12) and Lemmas 3.6 and 3.9 allow us to assume that $\tilde{S}$ is irreducible on V. Consequently 7.6, 7.7, 15.1.5, and 15.1.8 of [1] imply that

$$\tilde{S} \text{ is absolutely irreducible on } V. \tag{16}$$

Hence by Lemma 3.2 we can assume that

$$\begin{gathered}\text{the representation of } \tilde{S} \text{ in } GL(V) \\ \text{is defined over no proper subfield of } \mathbf{F}.\end{gathered} \tag{17}$$

LEMMA 3.10. *The group S is simple.*

Proof. Otherwise, Lemma 2.3(ii) implies that q is even and we write $S = S_1 \times \cdots \times S_k$, with S_i non-abelian and simple for each i and $k \geqslant 2$. Since $C_{GL(V)}(S_1)$ is not cyclic, S_1 is reducible on V. By Clifford's Theorem, there are 1, 2, or 4 homogeneous components of S_1 on V, permuted transitively by the perfect group S/S_1. Hence there is just one such component; that is, S_1 acts homogeneously on V. Since S is absolutely irreducible on V, it follows that S_1 is absolutely irreducible on the S_1-submodules of V (see 5.7 of [1], for example) and so S_1 satisfies the hypotheses of 10.3 of [1]. Thus S stabilizes a tensor product as described in the conclusion of 10.3 of [1]. But then by the proof of 15.1.12 of [1], S is contained in a group $K = Sp_6(q^3)$ acting irreducibly on V in its spin representation. But either K^τ or K^{τ^2} fixes a non-singular 1-space (see 15.1.3 of [1]), contrary to (16). ∎

We are now in a position to invoke the classification of finite simple groups. Namely, we seek to find all finite non-abelian simple subgroups S of H_0 such that $\tilde{S}$ satisfies (16) and (17).

S of Lie Type in Characteristic p

Let q_1 be an arbitrary power of p and put $\mathbf{F}_1 = GF(q_1)$. It is well known that none of the exceptional groups of type ${}^2B_2$, G_2, ${}^2G_2$, F_4, ${}^2F_4$, E_6, ${}^2E_6$, E_7, or E_8 have 8-dimensional absolutely irreducible p-modular representations (see Theorem 2.10 of [9], for example). Moreover, $|{}^3D_4(q_1)|$ divides $|H_0|$ only when $q = q_1^\alpha$ for some α, and as $GF(q_1^3)$ is a splitting field for ${}^3D_4(q_1)$, any proper subgroup ${}^3D_4(q_1)$ of H_0 cannot satisfy (17). Therefore we are left with the case in which S is classical.

First consider the case $S \cong L_2(q_1)$. According to the proof of 2.3.6 of [8], $q_1 \in \{q^3, q^9\}$. Obviously $L_2(q^9) \nleqslant H_0$, hence $q_1 = q^3$ and either (a) or (b) holds in the proof of 2.3.6 of [8], with q replaced by q^3. If (b) holds, then $\tilde{S}$ is contained in a tensor product group $Sp_2(q^3) \otimes Sp_4(q^3)$. But then by 15.1.6 of [1], S fixes a non-degenerate 3-space, a contradiction. If (a) holds, then the proof of 15.1.14 of [1] shows that S is contained in an irreducible $Sp_6(q^3)$ and we reach the same contradiction as in the proof of Lemma 3.10. Therefore $S \ncong L_2(q_1)$.

Next, if $S \cong L_m(q_1)$ $(m \geqslant 3)$, $PSp_{2m}(q_1)$ $(m \geqslant 2)$, $\Omega_{2m+1}(q_1)$ $(m \geqslant 3)$, or $P\Omega_{2m}^+(q_1)$ $(m \geqslant 4)$, then $\mathbf{F}_1$ is a splitting field for S and hence (17) implies that $\mathbf{F} \leqslant \mathbf{F}_1$. But then $|S|$ does not divide $|H_0|$. Similarly, if $S \cong U_m(q_1)$ $(m \geqslant 3)$ or $P\Omega_{2m}^-(q_1)$ $(m \geqslant 4)$, then $GF(q_1^2)$ is a splitting field and $\mathbf{F} \leqslant GF(q_1^2)$. Once again Lagrange's Theorem eliminates this case.

S Alternating, Sporadic, or of Lie Type in Characteristic Prime to p

We remarked at the end of Sect. 2.3 of [8] that none of the sporadic simple groups have an 8-dimensional irreducible p-modular representation. Thus we are left with the case where S is isomorphic to one of the groups appearing in (2*l*) of [8]. If S is isomorphic to $L_3(4)$, $L_4(2)$, $U_4(2)$, $Sp_6(2)$, or $\Omega_8^+(2)$, then p is odd because of our assumption on the characteristic. However, these groups all have 2-rank at least 4, contrary to Lemma 2.3(ii). Similarly $S \ncong U_4(3)$, for $m_3(U_4(3)) \geqslant 3$. If S is any of the remaining groups in (2*l*), then it follows from the ordinary and modular character table of S (see [4] and [10], for instance) that any absolutely 8-dimensional p-modular representation of S is writable over $GF(p^2)$. Hence S cannot satisfy (17), and we have now eliminated all possibilities for S.

The results in Section 3 imply

PROPOSITION 3.11. *If M_0 is non-local then M_0 appears on The List.*

4. Completion of the Proof

Propositions 2.4 and 3.11 guarantee that M_0 is conjugate to a group on The List. Thus putting $H = H_0$, we see that every maximal subgroup of H_0 appears on The List. Nothing more than elementary arguments is required to show that no group on The List is contained in any other; thus every group on The List is indeed maximal in H_0. As an example, we show that $M_0 = N_{H_0}(T_4)$ is maximal in H_0 for all q. Assume first that $q \neq 2$. By [12], there is a prime divisor r of $q^3+1 = p^{3n}+1$ such that r does not divide $p^m - 1$ for $1 \leqslant m \leqslant 6n-1$, and M_0 is a Sylow r-normalizer in H_0. Clearly no other group on The List contains a Sylow r-subgroup of H_0, save $N_{H_0}(\langle s_9 \rangle)$. However, a Sylow r-normalizer of $N_{H_0}(\langle s_9 \rangle)$ has order $(q^2-q+1)^2 \cdot 6$, and so $M_0 \nleqslant N_{H_0}(\langle s_9 \rangle)$. When $q = 2$, however, $M_0 \cong 3^2 \cdot SL_2(3)$ has order $2^3 3^3$ which divides the order of $C_{H_0}(g_1) \cong G_2(2)$, $N_{H_0}(F) \cong L_2(2) \times L_2(8)$ and $N_{H_0}(\langle s_9 \rangle) \cong SU_3(2)$. [6]. We show that M_0 is contained on none of these groups, as follows. First suppose that $M_0 \leqslant C_{H_0}(g_1)$. As M_0 has no subgroup of index 2, we have $M_0 \leqslant C_{H_0}(g_1)' \cong U_3(3)$. But then M_0 is the 3-local $3^{1+2} : 8$ in $U_3(3)$, which is impossible since M_0 has no elements of order 8. Second observe that $M_0 \nleqslant N_{H_0}(F)$, for the Sylow 3-subgroup of M_0 is non-abelian while that of $N_{H_0}(F)$ is abelian. Third suppose that $M_0 \leqslant N_{H_0}(\langle s_9 \rangle)$. Clearly M_0 has no normal subgroup of order 3, and so $s_9 \notin M_0$. But then $\langle s_9 \rangle \times O_3(M_0)$ is elementary abelian of order 27, contrary to Lemma 2.3(iii). Thus $N_{H_0}(T_4)$ is indeed maximal for all q, and it is left to the reader to show that the remaining groups on The List are maximal as well. The last assertion in the Theorem now follows. For suppose that L is a group on The List and that $N_H(L) \leqslant K < H$, with K maximal in H. Since any two isomorphic maximal subgroups of H_0 are conjugate in H_0, a Frattini argument yields $H = H_0 N_H(L)$. Therefore $H_0 \nleqslant K$, and so $K = N_H(K \cap H_0)$. However, $L \leqslant K \cap H_0 < H_0$, and the maximality of L ensures that $L = K \cap H_0$, which means $N_H(L) = K$, as desired. The proof of the Theorem is now complete.

Acknowledgments

This paper comprises part of the author's Ph.D. work, which was supervised by Dr. M. W. Liebeck and supported financially by Trinity College, Cambridge. The author would like to thank Dr. J. Saxl, Dr. R. A. Wilson, and especially Dr. Liebeck for consistent encouragement and valuable advice.

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